Sec 4.6 Related Rates

What is a Rate of Change?

change of output parameter \( \frac{d}{dt} \) an input parameter
rate inputs are often (not always) time \( \Rightarrow \) we'll focus on time
(example: surface area change wrt time \( \frac{d}{dt} \)

\[ \text{wrt} = \text{"with respect to"} \]

Formulas:

**Area:**
The radius \( r \) and area \( A \) of a circle are related by the equation \( A = \pi r^2 \).

Write an equation that relates \( \frac{dA}{dt} \) to \( \frac{dr}{dt} \).

\[ \frac{dA}{dt} = 2\pi r \frac{dr}{dt} \quad (1) \]

For ex. if \( r = 3 \text{ cm} \) and rate of radius change is \( 1 \text{ cm/sec} \) what is rate of area change?

Using rel (1) we have \( \frac{dA}{dt} = 2\pi (3) \cdot 1 = 6\pi \text{ cm}^2/\text{sec} \).

The side \( s \) and area \( A \) of a square are related by the equation \( A = s^2 \).

Write an equation that relates \( \frac{dA}{dt} \) to \( \frac{ds}{dt} \).

\[ \frac{dA}{dt} = 2s \frac{ds}{dt} \]

**Volume:**
The radius \( r \) and volume \( V \) of a sphere are related by the equation \( V = \frac{4}{3} \pi r^3 \).

Write an equation that relates \( \frac{dV}{dt} \) to \( \frac{dr}{dt} \).

\[ \frac{dV}{dt} = \frac{4}{3} \pi \cdot 3r^2 \frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt} \]

Ex 1. A balloon is filled with air. What is the relationship between the rate of change of volume and the rate of change of radius.

\[ \frac{dV}{dt} = \frac{4\pi \cdot 3r^2 \frac{dr}{dt}}{3} \quad \text{re-write as} \quad \frac{dV}{dt} = k \frac{dr}{dt} \]

Ex 2. What is the rate of increase of the volume of a balloon when the radius is 3 cm and the radius is increasing at a constant rate of 2 cm/sec.

\[ \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \]

\[ \frac{dV}{dt} = (4\pi (3)^2) \cdot 2 \]

\[ \frac{dV}{dt} = 72\pi \text{ cm}^3/\text{sec} \]
Ex 3. The radius of a circular disk is increasing at the rate of 3 cm/sec. What is the rate at which the area is changing when the radius is 4?

\[
\frac{dA}{dt} = 2\pi (r) \frac{dr}{dt} = 2\pi (4) \cdot 3 = 24\pi \text{ cm}^2/\text{sec}
\]

Ex 4. Find the rate of change of the area of a square when the sides have length 3 cm and the sides are increasing at the constant rate of 9 cm/sec.

\[
\frac{dA}{dt} = 2 \cdot l \frac{dl}{dt}
\]

\[
\frac{dA}{dt} = 2(3) \cdot 9 = 54 \text{ cm}^2/\text{sec}
\]

Ex 5. Water is being poured into a container in the shape of a cylinder 4 cm high and 2 cm in radius at a rate of 6 cm$^3$/sec. At what rate does the surface of the water rise?

\[
\frac{dV}{dt} = \pi r^2 h
\]

\[
\frac{dV}{dt} = \pi (2^2) \frac{dh}{dt} = \frac{6}{\pi} \cdot \frac{2^2}{\pi \cdot 2^2} = \frac{3}{2\pi} \text{ cm/sec}
\]

Note: it's plus because water is rising.

Ex 6. Water is being poured into a container in the shape of a rectangular box with square base (4 cm x 4 cm x 12 cm) at a rate of 50 cm$^3$/min. At what rate does the surface of the water rise?

\[
\frac{dV}{dt} = 50 \text{ cm}^3/\text{min}
\]

\[
\frac{dV}{dt} = S^2 \cdot h \frac{dh}{dt}
\]

\[
50 = 4^2 \frac{dh}{dt}
\]

\[
\frac{dh}{dt} = \frac{50}{16} = \frac{25}{8} \approx 3.13 \text{ cm/min}
\]
Ex 7. A conical reservoir is being filled with water at a constant rate of 6 m$^3$/min. If the reservoir is 3 m deep and 8 m in diameter at the top, find the rate at which the surface level of the water is rising at the instant the depth of the water is 2 m.

\[
\frac{dV}{dt} = 6 \text{ m}^3/\text{min}
\]
\[
h = 3 \text{ m}
\]
\[
r = 4 \text{ m}
\]
\[
\frac{h}{r} = \frac{3}{4}
\]

Now, let's consider the volume of the cone and its rate of change.

\[
V_{\text{cone}} = \frac{\pi r^2 h}{3}
\]
\[
\frac{dV}{dt} = \frac{16\pi h^2}{9} \frac{dh}{dt}
\]
\[
6 = \frac{16\pi h^2}{9} \frac{dh}{dt}
\]
\[
\frac{dh}{dt} = \frac{54}{16\pi(2)^2} = \frac{27}{32\pi} \text{ m/min}
\]

h = 2 \implies 0.27 \text{ m/min.}

Sometimes more than two quantities are changing within a certain relationship. At this point, you must consider the product rule, the quotient rule, and the chain rule when differentiating to express how different rates of change are related.

**Areas**

**Rectangle:**

\[
A = lw
\]

Find an equation that relates \( \frac{dA}{dt}, \frac{dl}{dt}, \text{ and } \frac{dw}{dt} \)

\[
\frac{dA}{dt} = \frac{dl}{dt} \cdot w + l \cdot \frac{dw}{dt}
\]

**Triangle:**

\[
A = \frac{1}{2} bh
\]

Find an equation that relates \( \frac{dA}{dt}, \frac{db}{dt}, \text{ and } \frac{dh}{dt} \)

\[
\frac{dA}{dt} = \frac{1}{2} \left[ b \frac{dh}{dt} + \frac{db}{dt} \cdot h \right]
\]

**Volumes**

**Cylinder:**

\[
V = \pi r^2 h
\]

Find an equation that relates \( \frac{dV}{dt}, \frac{dr}{dt}, \text{ and } \frac{dh}{dt} \)

\[
\frac{dV}{dt} = \pi \left[ 2\pi r \frac{dr}{dt} + r^2 \frac{dh}{dt} \right]
\]

**Cone:**

\[
V = \frac{1}{3} \pi r^2 h
\]

Find an equation that relates \( \frac{dV}{dt}, \frac{dr}{dt}, \text{ and } \frac{dh}{dt} \)

\[
\frac{dV}{dt} = \frac{1}{3} \pi \left( 2\pi r \frac{dr}{dt} + r^2 \frac{dh}{dt} \right)
\]

**Cone:**

\[
V = \frac{1}{3} \pi r^2 h
\]

Find an equation that relates \( \frac{dV}{dt} \) and \( \frac{dr}{dt} \) if \( h \) is a constant.
Sphere: \[ V = \frac{4}{3} \pi r^3 \] 
Find an equation that relates \( \frac{dV}{dt} \) and \( \frac{dr}{dt} \)

\[ \frac{dV}{dt} = \frac{4}{3} \pi \cdot 3r^2 \frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt} \]

Rectangular box: \[ V = lwh \] 
Find an equation that relates \( \frac{dV}{dt} \), \( \frac{dl}{dt} \), \( \frac{dw}{dt} \), and \( \frac{dh}{dt} \)

\[ \frac{dV}{dt} = lw \frac{dl}{dt} + lh \frac{dw}{dt} + wh \frac{dh}{dt} \]

Lengths of Sides

Right Triangle: \( x^2 + y^2 = z^2 \) 
Find an equation that relates \( \frac{dx}{dt} \), \( \frac{dy}{dt} \), and \( \frac{dz}{dt} \)

\[ 2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 2z \frac{dz}{dt} \]

Right Triangle: \( x^2 + y^2 = z^2 \) 
Find an equation that relates \( \frac{dx}{dt} \) and \( \frac{dz}{dt} \) if \( y \) is a constant.

\[ 2x \frac{dx}{dt} + 0 = 2z \frac{dz}{dt} \]

Diagonal of a rectangle: \( D = \sqrt{x^2 + y^2} \) 
Find an equation that relates \( \frac{dD}{dt} \), \( \frac{dx}{dt} \), and \( \frac{dy}{dt} \)

\[ \frac{dD}{dt} = \frac{2x \frac{dx}{dt} + 2y \frac{dy}{dt}}{\sqrt{x^2 + y^2}} \]

\[ \frac{dD}{dt} = \frac{x \frac{dx}{dt} + y \frac{dy}{dt}}{\sqrt{x^2 + y^2}} \]

Right Triangle: \( \tan \theta = \frac{y}{100} \) 
Find an equation that relates \( \frac{d\theta}{dt} \) and \( \frac{dy}{dt} \) if one side of this triangle is made constant.

\[ \sec^2 \theta \frac{d\theta}{dt} = \frac{1}{100} \frac{dy}{dt} \]