The Cone and Conic Sections

1) The Cone

Consider a right triangle with hypotenuse $c$, and legs $a$ and $b$. We can generate a 3–dimensional solid called a cone by rotating the triangle about its leg $b$. The leg $a$ generates a circle, which becomes the base of the cone.

The leg $c$ is called the generator of the cone.

The line $l$ is the axis of symmetry of the cone.

The point $A$ is called the vertex of the cone.

If we place together the vertices of two cones such that they share the same axis of symmetry, we have what is called a double-napped cone.

Label the diagram with the following.

1) vertex $A$
2) axis of symmetry
3) generator

2. Conic Sections

When a plane intersects a cone, it can form 2–dimensional objects called conic sections. They type of conic section formed depends upon the angle at which the plane intersects the cone.
Conic: **Circle**  
The plane is parallel to the base.

Conic: **Ellipse**  
The plane is inclined towards the base of the cone.

Conic: **Parabola**  
The plane is parallel to the generator of the cone.

Conic: **Hyperbola**  
The plane is parallel to the axis of symmetry, and intersects both nappes of the cone. A hyperbola has two distinct branches.
The Circle  Definition of a circle.

A circle is a set of all points (locus) in a plane that are equidistant from a fixed point

The fixed point is called the ________.
The distance from the centre to any point on the circle is called the ____________.

Ex 1: Write the equation of the circle with radius 5 and centre (3, 2).

Let $P(x,y)$ be any point on the circle.

Then, using the distance formula, we have

$$\sqrt{(x - 3)^2 + (y - 2)^2} = 5$$

The equation of the circle is

$$(x - 3)^2 + (y - 2)^2 = 25$$

2. The Standard Form of the Equation of a Circle

The following is the standard form of the equation of a circle with radius $r$, and centre $(h,k)$.

$$(x - h)^2 + (y - k)^2 = r^2$$

Ex 2: Write the equation of the circle with radius 7 and centre $(-4,8)$.

$$\left(x + 4\right)^2 + \left(y - 8\right)^2 = 49$$

3. The General Form of the Equation of a Circle

If we expand the equation in example 2, we obtain the following result.

For the circle in example 2, the equation is said to be in the general form.

The following is the general form of the equation of a circle.

$$Ax^2 + Ay^2 + Dx + Ey + F = 0$$

Thus for the circle in Example 2, we have $A = 1$, $D = 8$, $E = -16$ and $F = 31$.

Note: We are reserving the real constants $B$ and $C$ for the general forms of the ellipse, hyperbola, and parabola.

Ex 3: Determine the radius and centre of the circle with general equation $x^2 + 10x + y^2 - 6y + 18 = 0$
Note: To transform \( x^2 + 10x + y^2 - 6y + 18 = 0 \) into standard form, which tells us the centre and radius of the circle, we use the process called “completing the square.”

\[
(x^2 + 10x + 25) + (y^2 - 6y + 9) = -18 + 25 + 9
\]

\[
(x+5)^2 + (y-3)^2 = 16
\]

Centre: \( C(-5, 3) \)

Radius: \( r = 4 \)

**Ex 5:** Graph the equation \((x + 5)^2 + (y - 3)^2 = 16\) using a graphing calculator. (Use the Square viewing window.) Sketch the result on the grid below.

First, solve the equation for \( y \) in terms of \( x \).

\[
y - 3 = \pm \sqrt{16 - (x+5)^2}
\]

This means we must graph 2 functions:

- \( Y_1 = \sqrt{16 - (x+5)^2} + 3 \)
- \( Y_2 = -\sqrt{16 - (x+5)^2} + 3 \)

The Ellipse

An **Ellipse** is the set of points \( P \) such that the sum of the distances from two given points in the plane, the foci, is a constant. \( PF_1 + PF_2 = \) constant

**Standard Equations of Ellipses Centred at \((0, 0)\)**

- **Major axis on the \( x \)-axis.** \((a > b)\)
  
  \[
  \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1
  \]

  \( a = 4, \ b = 2 \)

- **Major axis on the \( y \)-axis.** \((a < b)\)
  
  \[
  \frac{x^2}{b^2} + \frac{y^2}{a^2} = 1
  \]

  \( a = 2, \ b = 4 \)

The squared terms are separated by a + sign. The term in which the larger denominator occurs indicates which axis contains the major axis. If \( a = b \), the equation represents a circle. Hence, a circle is a special case of an ellipse.
Ex 1 Name (Find) the conic section defined by the relation $9x^2 + 4y^2 = 36$, and sketch its graph.

\[
\begin{align*}
9\frac{x^2}{36} + 4\frac{y^2}{36} &= 1 \\
\frac{x^2}{4} + \frac{y^2}{9} &= 1 \\
\frac{x^2}{2^2} + \frac{y^2}{3^2} &= 1
\end{align*}
\]

**Standard Form** of an ellipse with centre $C(h, k)$ is

\[
\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1
\]

Major axis is parallel to $x$-axis. (if $a > b$) Length of major axis : $2a$ Length of minor axis : $2b$

Major axis is parallel to $y$-axis. (if $a < b$) Length of major axis : $2b$ Length of minor axis : $2a$

Ex 2 Describe and sketch each relation.

a) \[
\frac{(x+5)^2}{9} + \frac{(y-3)^2}{25} = 1 \quad \begin{align*}
a &= 3 \\
b &= 5
\end{align*}
\]

b) \[
\frac{(x-2)^2}{144} + \frac{(y+2)^2}{144} = 1 \quad \begin{align*}
a &= 4 \\
b &= 3
\end{align*}
\]
Graphing with a Calculator: Since circles and ellipses are not functions, it is necessary to use two functions to graph the relation. For example, the equation of the circle centred at the origin with radius $5$, $x^2 + y^2 = 25$, must first be solved for $y$ and then entered as $Y_1 = \sqrt{25 - x^2}$ and $Y_2 = -\sqrt{25 - x^2}$. It is important that you choose a square window.

Ex 3 Graph the ellipse \( \frac{(x+5)^2}{9} + \frac{(y-3)^2}{25} = 1 \) in a square window.

Ex 4. For the ellipse \( 9x^2 + 4y^2 + 36x - 64y + 256 = 0 \), determine
a) coordinates of the centre, and vertices
b) length of the major and minor axis
c) sketch the graph